Unit IV

Hyperbolic functions: Definition - Relation between hyperbolic function ~~and~~ inverse hyperbolic function.

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Hyperbolic function - Definition

Let x be a real (or) complex number,

(i) the quantity $\dfrac{e^x - e^{-x}}{2}$ is called hyperbolic sine of x

ii) the quantity $\dfrac{e^x + e^{-x}}{2}$ is called hyperbolic cosine of x

i.e, $\dfrac{e^x - e^{-x}}{2} = \sin hx$

$$\frac{e^x + e^{-x}}{2} = \cos hx$$

Other hyperbolic functions are,

$$\tan h\,x = \frac{\sin h\,x}{\cos h\,x}$$

$$= \frac{\left(\frac{e^x - e^{-x}}{2}\right)}{\left(\frac{e^x + e^{-x}}{2}\right)}$$

$$\boxed{\tan h\,x = \frac{e^x - e^{-x}}{e^x + e^{-x}}} \Rightarrow \boxed{\cot h\,x = \frac{e^x + e^{-x}}{e^x - e^{-x}}}$$

$$\operatorname{cosec} h\,x = \frac{1}{\sin hx} \quad \text{and} \quad \sec hx = \frac{1}{\cos hx}$$

$$\boxed{\operatorname{cosec} hx = \frac{2}{e^x - e^{-x}}} \quad \text{and} \quad \boxed{\sec hx = \frac{2}{e^x + e^{-x}}}$$

Points to Remember:

① $e^{x} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

② $e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$

③ $e^{ax} = 1 + \frac{(ax)}{1!} + \frac{(ax)^2}{2!} + \frac{(ax)^3}{3!} + \cdots$

④ $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

⑤ $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

Expanded form of $\sinh x$ and $\cosh x$

(i) we know that

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$= \frac{1}{2}\left[e^x - e^{-x}\right]$$

$$= \frac{1}{2}\left[\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) - \left(1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots\right)\right]$$

$$= \frac{1}{2}\left[\frac{2x}{1!} + 2\frac{x^3}{3!} + 2\frac{x^5}{5!} + \cdots\right]$$

$$\boxed{\sinh x = \frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots}$$

(ii) we know that,

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$= \frac{1}{2}\left[e^x + e^{-x}\right]$$

$$= \frac{1}{2}\left[\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) + 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots\right]$$

$$= \frac{1}{2}\left[2 + 2\frac{x^2}{2!} + 2\frac{x^4}{4!} + \cdots\right]$$

$$\boxed{\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots}$$

Relation between hyperbolic function and circular function

(i) we know that the circular functions

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad \text{and} \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

If we replace θ by ix, we get the hyperbolic function

$$\cos ix = \frac{e^{i(ix)} + e^{-i(ix)}}{2}, \qquad \sin ix = \frac{e^{i(ix)} - e^{-i(ix)}}{2i}$$

$$= \frac{e^{-x} + e^{x}}{2}, \qquad \sin ix = \frac{e^{-x} - e^{x}}{2i}$$

$$= \frac{e^{x} + e^{-x}}{2}, \qquad = -\left(\frac{e^{x} - e^{-x}}{2i}\right)$$

$$= \frac{i^2(e^{x} - e^{-x})}{2i}$$

$$= i\left(\frac{e^{x} - e^{-x}}{2}\right)$$

$$\boxed{\cos ix = \cosh x}$$

$$\boxed{\sec ix = \sech x}$$

$$\boxed{\sin ix = i\sinh x}$$

$$\boxed{\operatorname{cosec} ix = -i\operatorname{cosech} x}$$

(ii) By the definition of hyperbolic function of sine,

(a) $\sin h x = \dfrac{e^{x} - e^{-x}}{2}$

Replace x by $i\theta$ we get the circular function

$$\sin h (i\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2}$$

Multiplying both Nr and Dr of RHS by i,

$$= \frac{i\,(e^{i\theta} - e^{-i\theta})}{2i}$$

$$\boxed{\sin h (i\theta) = i \sin\theta}$$

(b) $\cos h x = \dfrac{e^{x} + e^{-x}}{2}$

Replace x by $i\theta$

$$\cos h (i\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\boxed{\cos h (i\theta) = \cos\theta}$$

(c) $\tan h x = \dfrac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$

Replace x by $i\theta$,

$$\tan h (i\theta) = \frac{e^{i\theta} - e^{-i\theta}}{e^{i\theta} + e^{-i\theta}}$$

$$= \frac{2i \sin\theta}{2 \cos\theta}$$

$$\boxed{\tan h (i\theta) = i \tan\theta}$$

Note (1) $\sin i\theta = i \sin h\theta$

(2) $\cos i\theta = \cos h\theta$

(3) $\tan i\theta = i \tan h\theta$

Result:

Hyperbolic functions are periodic functions

(i) $\sin h x$ remains unchanged when x is increased by any multiple of $2\pi i$.

$$\sin h(x+2n\pi i) = \sin h x$$

$\therefore$ $\sin h x$ is a periodic function with period $2\pi i$

(ii) $$\cos h(x+2n\pi i) = \cos h x$$

$\therefore$ $\cos h x$ is a periodic function with period $2\pi i$

(iii) $$\tan h(x+n\pi i) = \tan h x$$

$\therefore$ $\tan h x$ is a periodic function with period πi

Similarly, $\operatorname{cosec} h x$, $\sec h x$ are periodic with period $2\pi i$

and $\cot h x$ is also a periodic with period πi

$$\cosh x = \frac{e^{x}+e^{-x}}{2}$$

$$\sinh x = \frac{e^{x}-e^{-x}}{2}$$

$$\tan h x = \frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}$$

Hyperbolic function - Formulae

1. $\cos h^2 x - \sin h^2 x = 1$
2. $\sec h^2 x + \tan h^2 x = 1$
3. $\cot h^2 x - \operatorname{cosec} h^2 x = 1$
4. $\sin h(x+y) = \sin h x \cos h y + \cos h x \sin h y$
5. $\sin h(x-y) = \sin h x \cos h y - \cos h x \sin h y$
6. $\cos h(x+y) = \cos h x \cos h y + \sin h x \sin h y$
7. $\cos h(x-y) = \cos h x \cos h y - \sin h x \sin h y$
8. $\tan h(x+y) = \dfrac{\tan h x + \tan h y}{1+\tan h x \cdot \tan h y}$
9. $\tan h(x-y) = \dfrac{\tan h x - \tan h y}{1-\tan h x \tan h y}$

$\cos ix = \cos h x$

$\sin ix = i \sin h x$

$\tan ix = i \tan h x$

$\sec ix = \sec h x$

$\operatorname{cosec} ix = -i \operatorname{cosec} h x$

$\cot ix = -i \cot h x$

(10) $\sinh 2x = 2 \sinh x \cosh x$

(11) $\cosh 2x = \cosh^2 x + \sinh^2 x$

(12) $\cosh 2x = 2\cosh^2 x - 1$

(13) $\cosh 2x = 1 + 2\sinh^2 x$

(14) $\cosh 2x = \dfrac{1+\tanh^2 x}{1-\tanh^2 x}$

(15) $\tanh 2x = \dfrac{2\tanh x}{1+\tanh^2 x}$

(16) $\sinh 2x = \dfrac{2\tanh x}{1-\tanh^2 x}$

(17) $\sinh 3x = 3\sinh x + 4\sinh^3 x$

(18) $\cosh 3x = 4\cosh^3 x - 3\cosh x$

(19) $\tanh 3x = \dfrac{3\tanh x + \tanh^3 x}{1+3\tanh^2 x}$

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Example 4.1

Prove that $\tanh(x+y+z) = \dfrac{\tanh x + \tanh y + \tanh z + \tanh x \cdot \tanh y \cdot \tanh z}{1 + \tanh x \tanh y + \tanh y \cdot \tanh z + \tanh z \cdot \tanh x}$

Proof: W.K.T,

$$\tan(\alpha+\beta+\gamma) = \frac{\tan\alpha + \tan\beta + \tan\gamma - \tan\alpha \cdot \tan\beta \cdot \tan\gamma}{1 - \tan\alpha \cdot \tan\beta - \tan\beta \tan\gamma - \tan\gamma \cdot \tan\alpha}$$

Replace α by ix, β by iy and $\gamma = iz$

$$\tan(ix+iy+iz) = \frac{\tan ix + \tan iy + \tan iz - \tan ix \cdot \tan iy \cdot \tan iz}{1 - \tan ix \cdot \tan iy - \tan iy \tan iz - \tan iz \tan ix}$$

$$i \tanh(x+y+z) = \frac{i \tanh x + i \tanh y + i \tanh z - i\tanh x \cdot i \tanh y \cdot i \tanh z}{1 - (i \tanh x)(i \tanh y) - (i \tanh y)(i \tanh z) - (i \tanh z)(i \tanh x)}$$

$$\tanh(x+y+z) = \frac{\tanh x + \tanh y + \tanh z + \tanh x \tanh y \tanh z}{1 + \tanh x \tanh y + \tanh y \tanh z + \tanh z \tanh x}$$

Hence the proof.

$\boxed{\begin{matrix} i^2 = -1 \\ i^3 = -i \end{matrix}}$

Example 4.2

Separate into real and imaginary parts of $\sin(x+iy)$ and $\cos(x+iy)$

Solution

(i) WKT $\sin(x+iy) = \sin x \cos iy + \cos x \sin iy$

$= \sin x \cosh y + \cos x (i \sinh y)$

$\sin(x+iy) = \sin x \cosh y + i \cos x \sinh y$

$\therefore$ Real Part of $\sin(x+iy) = \sin x \cosh y$

Imaginary Part of $\sin(x+iy) = \cos x \sinh y$

(ii) WKT $\cos(x+iy) = \cos x \cos(iy) - \sin x \sin(iy)$

$= \cos x \cosh y - \sin x \, i \sinh y$

$\cos(x+iy) = \cos x \cosh y - i \sin x \sinh y$

$\therefore$ Real Part is $\cos x \cosh y$

Imaginary Part is $-\sin x \sinh y$

Example 4.3 Separate $\sinh(x+iy)$ and $\cosh(x+iy)$ into real and imaginary parts.

Solution

(i) $\sinh(x+iy) = \frac{i}{i}\sinh(x+iy)$ $\{\because i\sinh\theta = \sin i\theta$

$= \frac{1}{i}\left[i\sinh(x+iy)\right]$

$= \frac{1}{i}\left[\sin i(x+iy)\right]$

$= \frac{1}{i}\left[\sin(ix-y)\right]$

$= \frac{1}{i}\left[\sin(ix)\cos y - \cos(ix)\sin y\right]$ $\{\because \frac{1}{i} = -i$

$= -i\left[i\sinh x\cos y - \cosh x\sin y\right]$ $\{\because i^2 = -1$

$\sinh(x+iy) = \sinh x\cos y + i\cosh x\sin y$

$\therefore$ Real Part $= \sinh x\cos y$

Imaginary Part $= \cosh x\sin y$

Example 4.4 Prove that $(\cosh x + \sinh x)^n = \cosh nx + \sinh nx$ where n being a positive integer

Proof $(\cosh x + \sinh x)^n = \left(\frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2}\right)^n$

$= \left(\frac{2e^x}{2}\right)^n$

$= e^{nx}$ $\{\because e^\theta = \cosh\theta + \sinh\theta$

$= \cancel{\cosh n\theta + \sinh n\theta}$

$= \cosh nx + \sinh nx$

Example 4.5 If $u = \log \tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$ then prove that

(i) $\cosh u = \sec\theta$

(ii) $\sinh u = \tan\theta$

(iii) $\tanh u = \sin\theta$

Proof

Given $u = \log \tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$

$$e^u = \tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$$

$\left\{ \because \tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \cdot \tan B} \right.$

$$= \frac{\tan\frac{\pi}{4} + \tan\frac{\theta}{2}}{1 - \tan\frac{\pi}{4}\cdot\tan\frac{\theta}{2}}$$

$\left\{ \because \tan\frac{\pi}{4} = 1 \right.$

$$= \frac{1 + \tan\frac{\theta}{2}}{1 - \tan\frac{\theta}{2}}$$

$$= \frac{1 + \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}}{1 - \frac{\sin\theta/2}{\cos\theta/2}}$$

$$= \frac{\cos\theta/2 + \sin\theta/2}{\cos\theta/2 - \sin\theta/2}$$

$$= \frac{\cos\theta/2 + \sin\theta/2}{\cos\theta/2 - \sin\theta/2} \cdot \frac{\cos\theta/2 + \sin\theta/2}{\cos\theta/2 + \sin\theta/2}$$

$$= \frac{(\cos\theta/2 + \sin\theta/2)^2}{\cos^2\theta/2 - \sin^2\theta/2}$$

$$= \frac{\cos^2\theta/2 + \sin^2\theta/2 + 2\sin\theta/2\cos\theta/2}{\cos\theta}$$

$$= \frac{1 + \sin\theta}{\cos\theta}$$

$$= \frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta}$$

$$e^{u} = \sec\theta + \tan\theta \longrightarrow (1)$$

$$e^{-u} = \sec\theta - \tan\theta \longrightarrow (2)$$

By the definition of hyperbolic function

$$\cos hu = \frac{e^{u} + e^{-u}}{2}$$

$$= \frac{2\sec\theta}{2} \qquad \{\because (1)+(2)$$

$$\boxed{\cos hu = \sec\theta}$$

$$\sin hu = \frac{e^{u} - e^{-u}}{2}$$

$$= \frac{2\tan\theta}{2} \qquad \{\because (1)-(2)$$

$$\boxed{\sin hu = \tan\theta}$$

$$\tan hu = \frac{\sin hu}{\cos hu}$$

$$= \frac{\tan\theta}{\sec\theta}$$

$$= \frac{\sin\theta/\cos\theta}{1/\cos\theta}$$

$$\boxed{\tan hu = \sin\theta}$$

Example 4.6 Prove that $(1+\cosh x+\sinh x)^n = 2^n \cosh^n \frac{x}{2}\left[\cosh\frac{nx}{2}+\sinh\frac{nx}{2}\right]$

Proof:

$LHS = (1+\cosh x+\sinh x)^n$ $\quad\left\{\because \cosh x = \cos ix,\ \sinh x = \frac{1}{i}\sin ix\right.$

$$= \left(1+\cos ix+\frac{1}{i}\sin ix\right)^n$$

$$= \left[2\cos^2\left(\frac{ix}{2}\right)+\frac{1}{i}\,2\sin\left(\frac{ix}{2}\right)\cos\left(\frac{ix}{2}\right)\right]^n$$

$$= \left[2\cosh^2\left(\frac{x}{2}\right)+\frac{1}{i}\,2\,i\sinh\left(\frac{x}{2}\right)\cosh\left(\frac{x}{2}\right)\right]^n$$

$$= \left[2\cosh\left(\frac{x}{2}\right)\left(\cosh\left(\frac{x}{2}\right)+\sinh\left(\frac{x}{2}\right)\right)\right]^n$$

$$= 2^n\cosh^n\left(\frac{x}{2}\right)\left[\cosh\left(\frac{x}{2}\right)+\sinh\left(\frac{x}{2}\right)\right]^n$$

$$= 2^n\cosh^n\left(\frac{x}{2}\right)\left[\cosh\left(\frac{nx}{2}\right)+\sinh\left(\frac{nx}{2}\right)\right]$$

$= RHS$

Example 4.7

If $\tan\frac{x}{2} = \tanh\frac{x}{2}$, Prove that $\cos x\cdot\cosh x = 1$

Proof

Given $\tan\frac{x}{2} = \tanh\left(\frac{x}{2}\right)$

Squaring on both sides

$$\tan^2\frac{x}{2} = \tanh^2\left(\frac{x}{2}\right)$$

$$\Rightarrow \frac{1}{\tan^2\frac{x}{2}} = \frac{1}{\tanh^2\left(\frac{x}{2}\right)}$$

Componendo and dividendo rule

If $a:b = c:d$

then $\frac{a+b}{a-b} = \frac{c+d}{c-d}$

By Componendo and dividendo rule

$$\frac{1+\tan^2\frac{x}{2}}{1-\tan^2\frac{x}{2}} = \frac{1+\tanh^2\left(\frac{x}{2}\right)}{1-\tanh^2\left(\frac{x}{2}\right)}$$

$$\frac{1}{\cos x} = \cosh x \Rightarrow \boxed{\cos x\cdot\cosh x = 1}$$

Example 48 If $\sin(A+iB) = x+iy$, Prove that

(i) $\frac{x^2}{\cosh^2 B} + \frac{y^2}{\sinh^2 B} = 1$

(ii) $x^2 \operatorname{cosec}^2 A - y^2 \sec^2 A = 1$

Proof

Given:

$x+iy = \sin(A+iB)$

$= \sin A \cos iB + \cos A \sin iB$

$= \sin A \cosh B + \cos A \, i \sinh B$

$x+iy = \sin A \cosh B + i \cos A \sinh B$

Equating the real and imaginary parts

$x = \sin A \cosh B$ and $y = \cos A \sinh B$

$\frac{x}{\cosh B} = \sin A \rightarrow (1)$ and $\frac{y}{\sinh B} = \cos A \rightarrow (2)$

(i) Squaring and adding (1) & (2)

$$\frac{x^2}{\cosh^2 B} + \frac{y^2}{\sinh^2 B} = \sin^2 A + \cos^2 A$$

$$\boxed{\frac{x^2}{\cosh^2 B} + \frac{y^2}{\sinh^2 B} = 1}$$

(ii) From (1), ~~2~~

$$\frac{x}{\sin A} = \cosh B \rightarrow (3)$$

From (ii)

$$\frac{y}{\cos A} = \sinh B \rightarrow (4)$$

Squaring and Subtracting (3) and (4)

$$\frac{x^2}{\sin^2 A} - \frac{y^2}{\cos^2 A} = \cosh^2 B - \sinh^2 B$$

$$\frac{x^2}{\sin^2 A} - \frac{y^2}{\cos^2 A} = 1$$

$$\Rightarrow \boxed{x^2 \operatorname{cosec}^2 A - y^2 \sec^2 A = 1}$$

Example 4.9 $\sin(\theta + i\phi) = \tan\alpha + i\sec\alpha$ Show that $\cos 2\theta \cdot \cosh 2\phi = 3$

Solution

Given: $\tan\alpha + i\sec\alpha = \sin(\theta + i\phi)$

$$= \sin\theta \cos(i\phi) + \cos\theta \sin(i\phi)$$

$$\tan\alpha + i\sec\alpha = \sin\theta \cdot \cosh\phi + i\cos\theta \sinh\phi$$

Equating the real and imaginary Parts

$\tan\alpha = \sin\theta \cosh\phi$ and $\sec\alpha = \cos\theta \sinh\phi$

By Formula,

$$\sec^2\alpha - \tan^2\alpha = 1$$

$$\cos^2\theta \sinh^2\phi - \sin^2\theta \cosh^2\phi = 1$$

$$\left(\frac{1+\cos 2\theta}{2}\right)\left(\frac{\cosh 2\phi - 1}{2}\right) - \left(\frac{1-\cos 2\theta}{2}\right)\left(\frac{1+\cosh 2\phi}{2}\right) = 1$$

$$\left[\frac{\cancel{\cosh 2\phi} - 1 + \cos 2\theta \cosh 2\phi - \cancel{\cos 2\theta}}{4}\right] - \left[\frac{1 + \cancel{\cosh 2\phi} - \cancel{\cos 2\theta} - \cos 2\theta \cosh 2\phi}{4}\right] = 1$$

$$-2 + 2\cos 2\theta \cosh 2\phi = 4$$

$$2\cos 2\theta \cosh 2\phi = 6$$

$$\boxed{\cos 2\theta \cosh 2\phi = 3}$$

Example 4.10 Show that $\left(\frac{1+\tanh x}{1-\tanh x}\right)^3 = \cosh 6x + \sinh 6x$

Proof

$$LHS = \left(\frac{1+\tanh x}{1-\tanh x}\right)^3$$

$$= \left[\frac{1+\frac{\sinh x}{\cosh x}}{1-\frac{\sinh x}{\cosh x}}\right]^3$$

$$= \left[\frac{\cosh x + \sinh x}{\cosh x - \sinh x}\right]^3$$

$$= \left[\frac{\cosh x + \sinh x}{\cosh x - \sinh x} \cdot \frac{\cosh x + \sinh x}{\cosh x + \sinh x}\right]^3$$

$$= \left[\frac{(\cosh x + \sinh x)^2}{\cosh^2 x - \sinh^2 x}\right]^3$$

$$= (\cosh x + \sinh x)^6 \qquad \{\because \cosh^2 x - \sinh^2 x = 1$$

$$= \cosh 6x + \sinh 6x$$

$$= RHS$$

4.11

Identity (i) Prove that $\cosh^2 x - \sinh^2 x = 1$

Proof

W.K.T $\sin^2\theta + \cos^2\theta = 1$

Replace θ by ix we get

$$\sin^2(ix) + \cos^2(ix) = 1$$

$$(i \sinh x)^2 + (\cosh x)^2 = 1$$

$$i^2 \sinh^2 x + \cosh^2 x = 1 \qquad \{\because i^2 = -1$$

$$\boxed{\cosh^2 x - \sinh^2 x = 1}$$

(ii) Prove that $\operatorname{sech}^2 x + \tanh^2 x = 1$

Proof

W.K.T $\cosh^2 x - \sinh^2 x = 1$

Dividing both sides by $\cosh^2 x$, we get

$$1 - \frac{\sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x}$$

$$1 - \tanh^2 x = \operatorname{sech}^2 x$$

$$\Rightarrow \boxed{\operatorname{sech}^2 x + \tanh^2 x = 1}$$

Inverse Hyperbolic Functions

Let $y = \sinh x$, x is called the inverse hyperbolic sine of y and it can be written as $x = \sinh^{-1} y$.

Similarly we can define other inverse hyperbolic functions.

(i) $\sinh^{-1} x = \log\left(x + \sqrt{x^2+1}\right)$

(ii) $\cosh^{-1} x = \log\left(x + \sqrt{x^2-1}\right)$

(iii) $\tanh^{-1} x = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$

Proof (i) Let $\sinh^{-1} x = y$.

then $x = \sinh y$.

$$x = \frac{e^y - e^{-y}}{2}$$

$$2x = e^y - e^{-y}$$

$$2x = e^y - \frac{1}{e^y}$$

$$2x = \frac{e^{2y} - 1}{e^y}$$

$$2x e^y = e^{2y} - 1$$

$$e^{2y} - 2x e^y - 1 = 0$$

$$(e^y)^2 - 2x e^y - 1 = 0$$

$$p^2 - 2px - 1 = 0$$ (Take $e^y = p$

It is a quadratic form in 'p'

$$p = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$a=1$
$b=-2x$
$c=-1$

$$= \frac{2x \pm \sqrt{4x^2-4(1)(-1)}}{2(1)}$$

$$= \frac{2x \pm \sqrt{4x^2+4}}{2}$$

$$= \frac{2x \pm \sqrt{4(x^2+1)}}{2}$$

$$= \frac{2x \pm 2\sqrt{x^2+1}}{2}$$

$$= 2\left(\frac{x \pm \sqrt{x^2+1}}{2}\right)$$

$$p = x \pm \sqrt{x^2+1}$$

$$\Rightarrow e^y = x \pm \sqrt{x^2+1}$$

$$e^y = x + \sqrt{x^2+1}$$ ($\because$ ignoring the negative sign

Taking log on both sides, we get

$$y = \log\left(x+\sqrt{x^2+1}\right)$$

$$\boxed{\sinh^{-1} x = \log\left(x+\sqrt{x^2+1}\right)}$$

Similarly we can prove $\cosh^{-1} x$ and $\tanh^{-1} x$

Example

Prove that $\tanh^{-1}x + \tanh^{-1}y = \tanh^{-1}\left[\frac{x+y}{1+xy}\right]$

Proof:

Let $\tanh^{-1}x = \theta$ and $\tanh^{-1}y = \phi$

Then $x = \tanh\theta$ and $y = \tanh\phi$

$$\tanh(\theta+\phi) = \frac{\tanh\theta + \tanh\phi}{1+\tanh\theta\cdot\tanh\phi}$$

$$\tanh(\theta+\phi) = \frac{x+y}{1+xy}$$

$$\theta+\phi = \tanh^{-1}\left[\frac{x+y}{1+xy}\right]$$

$$\boxed{\tanh^{-1}x + \tanh^{-1}y = \tanh^{-1}\left[\frac{x+y}{1+xy}\right]}$$

Example $\cosh^{-1}x + \cosh^{-1}y = \cosh^{-1}\left(xy + \sqrt{x^2-1}\sqrt{y^2-1}\right)$

Proof

Let $\cosh^{-1}x = \theta$ and $\cosh^{-1}y = \phi$

$x = \cosh\theta$ and $y = \cosh\phi$

Now $\cosh(\theta+\phi) = \cosh\theta\cosh\phi + \sinh\theta\sinh\phi$

$$= \cosh\theta\cosh\phi + \sqrt{1-\cosh^2\theta}\cdot\sqrt{1-\cosh^2\phi}$$

$$\cosh(\theta+\phi) = xy + \sqrt{1-x^2}\cdot\sqrt{1-y^2}$$

$$\theta+\phi = \cosh^{-1}\left(xy + \sqrt{1-x^2}\cdot\sqrt{1-y^2}\right)$$

$$\boxed{\cosh^{-1}x + \cosh^{-1}y = \cosh^{-1}\left(xy + \sqrt{1-x^2}\sqrt{1-y^2}\right)}$$

Example $\sinh^{-1}x + \sinh^{-1}y = \sinh^{-1}\left(x\sqrt{1+y^2} + y\sqrt{1+x^2}\right)$

Proof

let $\sinh^{-1}x = \theta$ and $\sinh^{-1}y = \phi$

then $x = \sinh\theta$ $y = \sinh\phi$

Now,

$$\sinh(\theta+\phi) = \sinh\theta\cosh\phi + \cosh\theta\sinh\phi$$

$$= \sinh\theta\sqrt{1-\sinh^2\phi} + \sqrt{1-\sinh^2\theta}\ \sinh\phi$$

$$= x\sqrt{1-y^2} + \sqrt{1-x^2}\ y$$

$$\theta+\phi = \sinh^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$$

$$\boxed{\sinh^{-1}x + \sinh^{-1}y = \sinh^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)}$$